Int. J. Solids Structures Vol. 14, pp. 545–558
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0020-7683/78/0701-0545/$02.00/0

THE THERMODYNAMICS OF MAXWELLIAN MATERIALS

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(*Received* 18 *July* 1977; *in revised form* 2 *November* 1977; *received for publication* 12 *December* 1977)

Abstract—In this paper, we study the thermodynamics of a class of nonlinear dissipative materials frequently called Maxwellian materials. Using the Clausius–Duhem inequality, we establish the restrictions on the constitutive equations and show that in non-equilibrium situations the stress behaves in a manner similar to an internal state variable. We further show that the stress relaxation function for the material must satisfy a dissipation inequality, and that, if all equilibrium states of the material are asymptotically stable, the stress relaxation function is an odd function of the overstress $\sigma-\bar{\sigma}^*$, where $\bar{\sigma}^*$ is the equilibrium stress. In addition, we consider the instantaneous and equilibrium responses of the material and prove that in these situations both the stress and the temperature are deriveable from the internal energy function. Finally, we consider briefly an example of a specific consititutive model and close with a discussion of an alternative formulation of the general theory in which the roles of stress and temperature are interchanged.

1. INTRODUCTION

The constitutive assumption

$$\dot{\sigma}=\bar{E}(\epsilon,\sigma)\dot{\epsilon}+\bar{G}(\epsilon,\sigma) \tag{1.1}$$

has been used extensively to study one-dimensional wave propagation in non-linear dissipative materials such as viscoplastic and viscoelastic materials.† Materials characterized by (1.1) are frequently called *Maxwellian*‡ where σ is the stress, ϵ is the strain, $\bar{E}$ is the instantaneous stress–strain modulus and $\bar{G}$ is the stress relaxation function.

Although this material model has found wide use, we are unaware of any thermodynamical formulation which encompasses (1.1). Our aim in this paper is to develop such a formulation for nonconductors of heat within the mathematical framework of thermodynamics proposed by Coleman and Noll[5]. Crucial to the successful development of our theory is the appropriate selection of the set of material response functions. Relation (1.1) is merely a special case of the more general system of constitutive equations that we wish to study.

Our choice of response functions has been strongly influenced by the result in classical thermodynamics that there exists conjugate pairs of thermomechanical variables, i.e. stress and strain or temperature and entropy. For example, suppose that we postulate a response function for the internal energy. Then, in the classical theory, we see that such a function can only depend on the strain and entropy. Furthermore, two conjugate pairs of variables must exist in that stress is derivable from the internal energy by taking the partial derivative of the energy with respect to the strain, and the temperature is obtained when the partial with respect to the entropy is evaluated. In general, regardless of the thermodynamic potential used in the classical theory, it is *never* a function of both variables in a conjugate pair but rather only one from each of the pairs. Thus the stress and strain connot appear together in the list of independent variables appearing in such a thermodynamic potential function.

There is a great volume of sound physical observation causing us to adhere to these concepts and we do not wish to violate tradition; however, such reasoning applies only to the realm of equilibrium states covered by the classical theory. Under dynamical situations we feel free to set aside these precepts and permit both members of each conjugate pair to appear in the internal energy function. This point of view certainly appears to be consistent with the functional dependence stated in (1.1) where both stress and strain appear as independent variables.

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†For example, see the text by Cristescu[1] and the review articles by Herrmann[2] and Nunziato *et al.*[3].

‡Constitutive equations of the type (1.1) were first introduced by Maxwell[4] in a discussion of the kinetic theory of gases.

Thus, in this paper, we will consider constitutive equations in which the internal energy, the stress rate, and the temperature rate are functions of strain, entropy, stress, and temperature as well as the strain rate and entropy rate. Furthermore, in keeping with (1.1), we require that the stress-rate and temperature-rate relations be linear in the strain rate and the entropy rate. While strain, entropy, stress, and temperature all appear as independent variables in the internal energy, the role of the stress and temperature is distinct from that of the strain and entropy. In a sense, the strain and entropy serve to characterize the equilibrium states of the material, as would be expected from the classical theory; while the stress and temperature act as internal or hidden variables, governed by evolutionary equations, which describe departures from equilibrium. This interpretation of stress and temperature as internal state variables permits us to develop the present theory along the lines proposed previously by Coleman and Gurtin[6] and Bowen[7] in studies of materials with internal state variables.

In Section 2, we record our constitutive assumption for nonconductors and the governing field equations. In Section 3, we examine the restrictions imposed on the constitutive equations by the Clausius–Duhem inequality. These results show that three independent response functions must be prescribed which give the internal energy, the temperature, and the stress relaxation of the material solely as functions of strain, entropy, and stress. The stress-rate relation is shown to be similar in form to (1.1) where the moduli are determined by the prescribed internal energy and temperature response functions, and the stress relaxation function is restricted by a dissipation inequality. The existence of a response function for the temperature eliminates the need for the temperature-rate relation.

The concept of equilibrium and the stability of equilibrium states is treated in Section 4. Here we show that the variables strain and entropy are sufficient to characterize the equilibrium states in a manner consistent with classical thermodynamics; and that at equilibrium, stress and temperature are derivable from the internal energy. Furthermore, we find that the internal energy is a minimum at all equilibrium states which are asymptotically stable. In Section 5 we make use of our equilibrium results to establish further restrictions on the stress relaxation function. In particular, *if all equilibrium states are asymptotically stable, then the stress relaxation function must be an odd function of the overstress* $\sigma - \bar{\sigma}^*$, where $\bar{\sigma}^*$ is the equilibrium value of the stress.

The instantaneous response of the material is covered in Section 6, and assuming that this response is elastic, we find that the stress and temperature are again deriveable from the internal energy function. In Section 7, we illustrate how the results of the theory can be used to construct a specific consitutive model.

Finally, in Section 8 we discuss an alternative formulation of the theory in which the roles of stress and temperature are interchanged so that the temperature is now governed by a rate relation. In this case the internal energy, the stress, and the temperature relaxation of the material are functions only of strain, entropy, and temperature. This formulation is shown to be entirely equivalent to the original formulation.

2. EQUATIONS OF MOTION AND MATERIAL RESPONSE

In this paper, we are interested in one-dimensional motions of a class of nonlinear, dissipative materials. Thus, it is convenient to identify the body in its reference configuration with an interval $\mathcal{B}$ of the real line $\mathcal{R} = (-\infty, \infty)$ and identify each material point with its position X in $\mathcal{B}$. In its reference configuration, the body is assumed to be homogeneous and have a density denoted by ρ_0. The motion of the body $\mathcal{B}$ is described by the continuous function $\hat{\chi}$ such that

$$x = \hat{\chi}(X, t) \tag{2.1}$$

gives the spatial position x at time $t \in \mathcal{R}$ of the particle which occupied the position X in the reference configuration. The displacement u of the material point X is defined by

$$u = \hat{\chi}(X, t) - X. \tag{2.2}$$

Assuming that the motion $\hat{\chi}$ is of class C^3, we can also define the strain (or displacement gradient) ϵ by

$$\epsilon = u_X. \tag{2.3}$$

Hereafter, we shall call the pair (ϵ, η) an *admissible strain-entropy pair* if both the strain ϵ and the entropy η are of class C^2 for all (X, t) in $\mathcal{B} \times \mathcal{R}$.

We now wish to consider constitutive equations which will characterize the thermomechanical response of dissipative materials of the type (1.1). In formulating these constitutive equations, we have kept in mind our introductory remarks on conjugate variables and included both mechanical and thermal relaxation effects. Consequently, we consider materials which do not conduct heat and are described by the constitutive equations†

$$e = \hat{e}(\epsilon, \eta, \sigma, \theta, \dot{\epsilon}, \dot{\eta}), \tag{2.4}$$

$$\dot{\sigma} = \hat{E}(\epsilon, \eta, \sigma, \theta)\dot{\epsilon} + \hat{F}(\epsilon, \eta, \sigma, \theta)\dot{\eta} + \hat{G}(\epsilon, \eta, \sigma, \theta), \tag{2.5}$$

$$\dot{\theta} = \hat{A}(\epsilon, \eta, \sigma, \theta)\dot{\epsilon} + \hat{B}(\epsilon, \eta, \sigma, \theta)\dot{\eta} + \hat{C}(\epsilon, \eta, \sigma, \theta), \tag{2.6}$$

where e is the internal energy per unit volume. The functions $\hat{E}$, $\hat{F}$, $\hat{A}$, and $\hat{B}$ are called the *instantaneous stress–strain modulus*, the *instantaneous stress-entropy modulus*, the *instantaneous temperature-strain modulus*, and the *instantaneous temperature-entropy modulus*, respectively. Throughout this paper, we assume that for all $(\epsilon, \eta, \sigma, \theta)$

$$\hat{E} \neq 0, \quad \hat{F} \neq 0, \quad \hat{A} \neq 0, \quad \hat{B} \neq 0. \tag{2.7}$$

The functions $\hat{G}$ and $\hat{C}$ appearing in (2.5) and (2.6) are called the *stress relaxation function* and the *temperature relaxation function*, respectively.

In order for this constitutive assumption to be physically meaningful, we need to ensure that the evolutionary equations (2.5) and (2.6) for the stress σ and temperature θ always have a solution. Therefore, we shall further require that for every admissible pair (ϵ, η) on $\mathcal{B} \times \mathcal{R}$ and initial values $\sigma(X, 0) = \sigma_0$, $\theta(X, 0) = \theta_0$, the differential equations (2.5) and (2.6) have class C^1 solutions on $\mathcal{B} \times \mathcal{R}$. Materials of the type (2.4)–(2.6) will be referred to as *generalized Maxwellian materials*.

In the present context, a *thermodynamic process* is the ordered array $\{u, \eta, \sigma, \theta, e\}$ such that $\hat{e}$ is of class C^2; $\hat{E}$, $\hat{F}$, $\hat{A}$, and $\hat{B}$ are of class C^1; and the balance laws

$$\rho_0 \ddot{u} = \sigma_X + b, \tag{2.8}$$

$$\dot{e} = \sigma\dot{\epsilon} + r, \tag{2.9}$$

along with (2.5) and (2.6), are satisfied on $\mathcal{B} \times \mathcal{R}$ for appropriate values of the body force b and heat supply r. For a given process to be *admissible* it must also satisfy the Clausius–Duhem inequality

$$\theta\dot{\eta} \geq r. \tag{2.10}$$

3. CONSEQUENCES OF THE CLAUSIUS–DUHEM INEQUALITY

Here we require that every thermodynamic process in a generalized Maxwellian material be an admissible process and consequently, the dissipation inequality (2.10) serves to impose restrictions on the form of the constitutive equations (2.4)–(2.6). These restrictions are contained in:

Theorem 1

Every thermodynamic process in a generalized Maxwellian material satisfies the Clausius–

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†These constitutive equations are consistent with the concept of equipresence; see Truesdell and Toupin[8].

Duhem inequality (2.10) *if and only if*

(i) $\hat{e}$ *is independent of* $\dot{\epsilon}$ *and* $\dot{\eta}$; i.e.

$$e = \hat{e}(\epsilon, \eta, \sigma, \theta), \tag{3.1}$$

(ii) $\hat{e}$, $\hat{E}$, $\hat{F}$, $\hat{A}$, *and* $\hat{B}$ *are related by*

$$(\hat{e}_\epsilon) + (\hat{e}_\sigma)\hat{E} + (\hat{e}_\theta)\hat{A} - \sigma = 0, \tag{3.2}$$

$$(\hat{e}_\eta) + (\hat{e}_\sigma)\hat{F} + (\hat{e}_\theta)\hat{B} - \theta = 0, \tag{3.3}$$

(iii) $\hat{e}$, $\hat{G}$, *and* $\hat{C}$ *obey the inequality*

$$(\hat{e}_\sigma)\hat{G} + (\hat{e}_\theta)\hat{C} \leq 0. \tag{3.4}$$

Proof

The proof of this result follows a now standard argument (see Coleman and Noll[5]). By the smoothness of the internal energy response function $\hat{e}$, (2.4), (2.5), (2.6), and the energy equation (2.9) imply that

$$\begin{aligned}\{(\hat{e}_\epsilon) + (\hat{e}_\sigma)\hat{E} + (\hat{e}_\theta)\hat{A} - \sigma\}\dot{\epsilon} + \{(\hat{e}_\eta) + (\hat{e}_\sigma)\hat{F} + (\hat{e}_\theta)\hat{B}\}\dot{\eta} + (\hat{e}_\sigma)\hat{G} + (\hat{e}_\theta)\hat{C}\\ + (\hat{e}_{\dot{\epsilon}})\ddot{\epsilon} + (\hat{e}_{\dot{\eta}})\ddot{\eta} = r.\end{aligned} \tag{3.5}$$

This form of the energy equation combines with (2.10) to yield the inequality

$$\begin{aligned}\{(\hat{e}_\epsilon) + (\hat{e}_\sigma)\hat{E} + (\hat{e}_\theta)\hat{A} - \sigma\}\dot{\epsilon} + \{(\hat{e}_\eta) + (\hat{e}_\sigma)\hat{F} + (\hat{e}_\theta)\hat{B} - \theta\}\dot{\eta} + (\hat{e}_\sigma)\hat{G} + (\hat{e}_\theta)\hat{C}\\ + (\hat{e}_{\dot{\epsilon}})\ddot{\epsilon} + (\hat{e}_{\dot{\eta}})\ddot{\eta} \leq 0.\end{aligned} \tag{3.6}$$

Clearly, for any point in $\mathscr{R} \times \mathscr{R}$ there exists an infinite set of thermodynamic processes corresponding to an admissible pair (ϵ, η). For such processes the values of $\dot{\epsilon}$, $\dot{\eta}$, $\ddot{\epsilon}$, and $\ddot{\eta}$ can vary independently of any of the other quantities in the inequality (3.6). Consequently, the coefficients, of $\dot{\epsilon}$, $\dot{\eta}$, $\ddot{\epsilon}$, and $\ddot{\eta}$ must vanish and (i)–(iii) of Theorem 1 follow directly. The sufficiency of (i)–(iii) is obvious from (3.6).

It should be noted that with the results of Theorem 1, (3.5) reduces to a more convenient form of the energy equation, i.e.

$$\theta\dot{\eta} + (\hat{e}_\sigma)\hat{G} + (\hat{e}_\theta)\hat{C} = r. \tag{3.7}$$

The results (3.1)–(3.3) have an important implication. In particular, they assert that there exists a class C^1 function $\hat{f}$ such that†

$$\begin{aligned}\hat{f}(\epsilon, \eta, \sigma, \theta) &= (\hat{e}_\epsilon) + (\hat{e}_\sigma)\hat{E} + (\hat{e}_\theta)\hat{A} - \sigma\\ &= (\hat{e}_\eta) + (\hat{e}_\sigma)\hat{F} + (\hat{e}_\theta)\hat{B} - \theta = 0.\end{aligned} \tag{3.8}$$

Thus, by the chain-rule,

$$(\hat{f}_\epsilon)\dot{\epsilon} + (\hat{f}_\eta)\dot{\eta} + (\hat{f}_\sigma)\dot{\sigma} + (\hat{f}_\theta)\dot{\theta} = 0. \tag{3.9}$$

Either (2.5) or (2.6) can be substituted into (3.9) and the complimentary relation (for either $\dot{\theta}$ or

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†We require that the solutions of (3.2) and (3.3) lie on the same hypersurface. If this were not true, (3.2) and (3.3) could be solved for σ and θ in terms of ϵ and η. Hence, the theory would degenerate to a dependence of the response functions on ϵ and η alone. The material response would be elastic in this case.

$\dot{\sigma}$) can be obtained provided $\hat{f}$ has the following properties:

$$(\hat{f}_\sigma) \neq 0, \quad (\hat{f}_\theta) \neq 0, \tag{3.10}$$

$$(\hat{f}_\epsilon) + (\hat{f}_\sigma)\hat{E} + (\hat{f}_\theta)\hat{A} = 0, \tag{3.11}$$

$$(\hat{f}_\eta) + (\hat{f}_\sigma)\hat{F} + (\hat{f}_\theta)\hat{B} = 0, \tag{3.12}$$

$$(\hat{f}_\sigma)\hat{G} + (\hat{f}_\theta)\hat{C} = 0. \tag{3.13}$$

With $(\hat{f}_\theta) \neq 0$ everywhere, (3.8) implies that there exists a function $\bar{\theta}$ such that

$$\theta = \bar{\theta}(\epsilon, \eta, \sigma)$$

and this relation is equivalent to the differential form (2.6)†. In view of this equivalence, we can reformulate the constitutive equations (2.4)–(2.6) and, in particular, we can prove:

Theorem 2

A generlized Maxwellian material can be characterized by the constitutive equations

$$e = \bar{e}(\epsilon, \eta, \sigma), \tag{3.14}$$

$$\theta = \bar{\theta}(\epsilon, \eta, \sigma), \tag{3.15}$$

$$\dot{\sigma} = \bar{E}(\epsilon, \eta, \sigma)\dot{\epsilon} + \bar{F}(\epsilon, \eta, \sigma)\dot{\eta} + \bar{G}(\epsilon, \eta, \sigma), \tag{3.16}$$

subject to the dissipation inequality

$$\bar{e}_\sigma(\epsilon, \eta, \sigma)\bar{G}(\epsilon, \eta, \sigma) \leq 0. \tag{3.17}$$

Furthermore, the response functions $\bar{E}$ and $\bar{F}$ are not arbitrary, but are determined by $\bar{e}$ and $\bar{\theta}$ through the relations

$$\bar{E}(\epsilon, \eta, \sigma)\bar{e}_\sigma(\epsilon, \eta, \sigma) = \sigma - \bar{e}_\epsilon(\epsilon, \eta, \sigma), \tag{3.18}$$

$$\bar{F}(\epsilon, \eta, \sigma)\bar{e}_\sigma(\epsilon, \eta, \sigma) = \bar{\theta}(\epsilon, \eta, \sigma) - \bar{e}_\eta(\epsilon, \eta, \sigma). \tag{3.19}$$

Proof

To prove these results, we first note that since (3.15) exists, (3.1) becomes

$$\begin{aligned} e = \hat{e}(\epsilon, \eta, \sigma, \theta) &= \hat{e}(\epsilon, \eta, \sigma, \bar{\theta}(\epsilon, \eta, \sigma)) \\ &= \bar{e}(\epsilon, \eta, \sigma) \end{aligned} \tag{3.20}$$

which is (3.14). Similarly, we can obtain the functions $\bar{E}$, $\bar{F}$, and $\bar{G}$ in (3.16). To obtain (3.17), we note that by (3.20),

$$\bar{e}_\sigma = \hat{e}_\sigma + \hat{e}_\theta\bar{\theta}_\sigma. \tag{3.21}$$

But, by (3.8) and (3.13),

$$\bar{\theta}_\sigma = -\frac{(\hat{f}_\sigma)}{(\hat{f}_\theta)} = \frac{\hat{C}}{\hat{G}}. \tag{3.22}$$

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†Since $\hat{f}_\sigma$ is also non-vanishing, there is an alternative formulation in which (2.5) is replaced by the function $\sigma = \hat{\sigma}(\epsilon, \eta, \theta)$. This formulation is discussed further in Section 8. It should also be noted that since both $\hat{f}_\theta$ and $\hat{f}_\sigma$ are non-vanishing, (3.13) asserts that if $\hat{G}$ or $\hat{C}$ vanish, then they vanish simultaneously.

Therefore, we see that, with $\hat{G} = \bar{G}$,

$$(\bar{e}_\sigma)\bar{G} = (\hat{e}_\sigma)\hat{G} + (\hat{e}_\theta)\hat{C} \tag{3.23}$$

and hence, (3.17) follows directly from (3.4). Finally, to establish (3.18) and (3.19), we need to compute the partial derivatives $\bar{e}_\epsilon$ and $\bar{e}_\eta$. By (3.20),

$$\bar{e}_\epsilon = \hat{e}_\epsilon + (\hat{e}_\theta)\bar{\theta}_\epsilon, \quad \bar{e}_\eta = \hat{e}_\eta + (\hat{e}_\theta)\bar{\theta}_\eta \tag{3.24}$$

where, by (3.8), (3.11)–(3.13), and (3.22),

$$\bar{\theta}_\epsilon = -\frac{(\hat{f}_\epsilon)}{(\hat{f}_\theta)} = \hat{E}\left(\frac{\hat{C}}{\hat{G}}\right) + \hat{A}, \tag{3.25}$$

$$\bar{\theta}_\eta = -\frac{(\hat{f}_\eta)}{(\hat{f}_\theta)} = \hat{F}\left(\frac{\hat{C}}{\hat{G}}\right) + \hat{B}. \tag{3.26}$$

Substituting (3.25) and (3.26) in (3.24) and using the resulting expressions with (3.2), (3.3), and (3.23), we can arrive at the desired results.

It is important to emphasize that in formulating thermodynamically consistent constitutive equations for generalized Maxwellian materials, three response functions, $\bar{e}$, $\bar{\theta}$ and $\bar{G}$, must be specified. This is more than is usually required in other dissipative theories, such as materials with memory where only one function need be specified, the internal energy (see Coleman[9]), and materials with internal state variables where two response functions are required, the internal energy and the vector rate equation for the internal variables[6, 7]. Again we remark that the present theory bears a striking similarity with the theory of materials with internal state variables with the stress σ acting as the internal variable. Consequently, much of our subsequent development of the theory of generalized Maxwellian materials will draw heavily on the ideas developed previously in studies of materials with internal variables (see Coleman and Gurtin[6], and Bowen[7]).

Finally, we observe that, by (3.23), the energy equation (3.7) becomes

$$\bar{\theta}(\epsilon, \eta, \sigma)\dot{\eta} + \bar{e}_\sigma(\epsilon, \eta, \sigma)\bar{G}(\epsilon, \eta, \sigma) = r. \tag{3.27}$$

4. EQUILIBRIUM STATES AND ASYMPTOTIC STABILITY

In dissipative materials, the notion of an equilibrium state is an extremely important one and here we discuss the properties of such states in the context of the theory of generalized Maxwellian materials.

In view of (3.16), we shall call a state $(\epsilon^*, \eta^*, \sigma^*)$ an *equilibrium state* (see Greenberg[10]) if

$$\bar{G}(\epsilon^*, \eta^*, \sigma^*) = 0. \tag{4.1}$$

Clearly then, the dissipation inequality (3.17) asserts that at a material point X in $\mathscr{B}$

$$\delta(t) = \bar{\delta}(\epsilon, \eta, \sigma) = \bar{e}_\sigma(\epsilon, \eta, \sigma)\bar{G}(\epsilon, \eta, \sigma) \tag{4.2}$$

is a maximum at equilibrium. Consequently,

$$\bar{\delta}_\epsilon(\epsilon^*, \eta^*, \sigma^*) = \bar{\delta}_\eta(\epsilon^*, \eta^*, \sigma^*) = \bar{\delta}_\sigma(\epsilon^*, \eta^*, \sigma^*) = 0 \tag{4.3}$$

and

$$\bar{\delta}_{\epsilon\epsilon}(\epsilon^*, \eta^*, \sigma^*) \leq 0, \quad \bar{\delta}_{\eta\eta}(\epsilon^*, \eta^*, \sigma^*) \leq 0, \quad \bar{\delta}_{\sigma\sigma}(\epsilon^*, \eta^*, \sigma^*) \leq 0. \tag{4.4}$$

Using these conditions, it is a simple matter to show that

$$\begin{aligned}(\bar{e}_\sigma)^*(\tilde{G}_\epsilon)^* &= 0,\\ (\bar{e}_\sigma)^*(\tilde{G}_\eta)^* &= 0,\\ (\bar{e}_\sigma)^*(\tilde{G}_\sigma)^* &= 0,\end{aligned} \tag{4.5}$$

and

$$2(\bar{e}_{\sigma\epsilon})^*(\tilde{G}_\epsilon)^* + (\bar{e}_\sigma)^*(\tilde{G}_{\epsilon\epsilon})^* \leq 0, \tag{4.6}$$

$$2(\bar{e}_{\sigma\eta})^*(\tilde{G}_\eta)^* + (\bar{e}_\sigma)^*(\tilde{G}_{\eta\eta})^* \leq 0, \tag{4.7}$$

$$2(\bar{e}_{\sigma\sigma})^*(\tilde{G}_\sigma)^* + (\bar{e}_\sigma)^*(\tilde{G}_{\sigma\sigma})^* \leq 0, \tag{4.8}$$

where we have used the notation ( )* to indicate that the function is evaluated at equilibrium. The results (4.5)–(4.8) clearly give an indication of the restrictions the dissipation inequality imposes on the response functions $\bar{e}$ and $\tilde{G}$ in the neighborhood of an equilibrium state. However, even stronger results hold if the equilibrium state is stable in a certain sense.

Consider again an equilibrium state $(\epsilon^*, \eta^*, \sigma^*)$ of the material at the point $X \in \mathcal{B}$. The *domain of attraction for the constant strain-entropy pair* (ϵ^*, η^*) of an equilibrium state is the set $\mathcal{D}^*$ of all values σ_0 such that the solution $\sigma = \sigma(t)$ of the initial value problem

$$\dot{\sigma} = \tilde{G}(\epsilon^*, \eta^*, \sigma), \tag{4.9}$$

$$\sigma(t_0) = \sigma_0, \tag{4.10}$$

exists for all time $t \geq t_0$, is unique, and

$$\sigma(t) \to \sigma^* \quad \text{as} \quad t \to \infty. \tag{4.11}$$

The equilibrium state $(\epsilon^*, \eta^*, \sigma^*)$ is said to be *asymptotically stable for the constant strain-entropy pair* (ϵ^*, η^*) if the set $\mathcal{D}^*$ contains a neighborhood of σ^*.

Now, it follows from (3.14), (3.16), and the chain-rule that at X there exists at least one thermodynamically admissible process corresponding to a constant strain-entropy pair (ϵ^*, η^*) such that

$$\dot{e} = \bar{e}_\sigma(\epsilon^*, \eta^*, \sigma)\tilde{G}(\epsilon^*, \eta^*, \sigma).$$

Furthermore, by the dissipation inequality (3.17), $e(t) \leq e(t_0)$ for all $t \geq t_0$, or, by (3.14) and (4.10),

$$\bar{e}(\epsilon^*, \eta^*, \sigma_0) \geq \bar{e}(\epsilon^*, \eta^*, \sigma(t)). \tag{4.12}$$

Since σ_0 is in $\mathcal{D}^*$, (4.11) and the smootheness of $\bar{e}$ implies that

$$\lim_{t \to \infty} \bar{e}(\epsilon^*, \eta^*, \sigma(t)) = \bar{e}(\epsilon^*, \eta^*, \sigma^*),$$

and consequently,

$$\bar{e}(\epsilon^*, \eta^*, \sigma_0) \geq \bar{e}(\epsilon^*, \eta^*, \sigma^*). \tag{4.13}$$

This result, along with the definition of asymptotic stability, yields the following minimal property of the internal energy.†

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†This minimal property also arises in other dissipative theories. See Coleman[9] who considered materials with fading memory and Coleman and Gurtin[6] who consider materials with internal state variables.

Theorem 3

At an equilibrium state which is asymptotically stable for the constant strain-entropy pair (ϵ^*, η^*), *the internal energy is a minimum*; i.e.

$$\tilde{e}(\epsilon^*, \eta^*, \sigma) \geq \tilde{e}(\epsilon^*, \eta^*. \sigma^*) \tag{4.14}$$

for all values of the stress σ *in some neighborhood of* σ^*. *Furthermore, at such an equiilibrium state*

$$\tilde{e}_\sigma(\epsilon^*, \eta^*, \sigma^*) = 0, \quad \tilde{e}_{\sigma\sigma}(\epsilon^*, \eta^*, \sigma^*) \geq 0, \tag{4.15}$$

and

$$\tilde{G}_\sigma(\epsilon^*, \eta^*, \sigma^*) \leq 0. \tag{4.16}$$

Clearly (4.15) is a consequence of (4.14) and (4.16) follows from (4.15) and (4.8).

From classical arguments, we expect that at equilibrium states a unique relationship exists which determines the stress σ^* in terms of the admissible pairs (ϵ^*, η^*):

$$\sigma^* = \tilde{\sigma}^*(\epsilon^*, \eta^*). \tag{4.17}$$

We can guarantee the existence of such a functional form by requiring the strict form of the inequality (4.16), i.e.

$$\tilde{G}_\sigma(\epsilon^*, \eta^*, \sigma^*) < 0, \tag{4.18}$$

to hold for all equilibrium states. Clearly, by (3.14) and (3.15), we also have

$$e^* = \tilde{e}(\epsilon^*, \eta^*, \tilde{\sigma}^*(\epsilon^*, \eta^*)) = \tilde{e}^*(\epsilon^*, \eta^*), \tag{4.19}$$

$$\theta^* = \tilde{\theta}(\epsilon^*, \eta^*, \tilde{\sigma}^*(\epsilon^*, \eta^*)) = \tilde{\theta}^*(\epsilon^*, \eta^*). \tag{4.20}$$

We call $\tilde{\sigma}^*$, $\tilde{e}^*$ and $\tilde{\theta}^*$ *the equilibrium response* functions for the material. Then, by the chain rule,

$$\tilde{e}^*_\epsilon = (\tilde{e}_\epsilon)^* + (\tilde{e}_\sigma)^* \tilde{\sigma}^*_\epsilon, \tag{4.21}$$

$$\tilde{e}^*_\eta = (\tilde{e}_\eta)^* + (\tilde{e}_\sigma)^* \tilde{\sigma}^*_\eta, \tag{4.22}$$

and using (3.18), (3.19) and (4.15), we have:

Theorem 4

If the equilibrium states of a generalized Maxwellian material are asymptotically stable and (4.18) *holds, then the equilibrium response of the material is elastic and characterized by the response function*

$$e = \tilde{e}^*(\epsilon, \eta) \tag{4.23}$$

where the stress σ *and the temperature* θ *are given by the thermodynamic relations*

$$\tilde{\sigma}^* = \tilde{e}^*_\epsilon, \quad \tilde{\theta}^* = \tilde{e}^*_\eta. \tag{4.24}$$

We conclude this section by establishing one other result of interest which has a bearing on the speed of propagation of weak disturbances in generalized Maxwellian materials.

Theorem 5

Consider a generalized Maxwellian material whose equilibrium states are asymptotically *stable. Further suppose that for every equilibrium state* $(\epsilon^*, \eta^*, \sigma^*)$,

$$\tilde{G}_\sigma(\epsilon^*, \eta^*, \sigma^*) < 0, \tag{4.25}$$

$$\tilde{G}_\epsilon(\epsilon^*, \eta^*, \sigma^*) > 0. \tag{4.26}$$

Then, the instantaneous stress–strain modulus $\tilde{E}$ *evaluated at the equilibrium state* $(\epsilon^*, \eta^*, \sigma^*)$ *is always greater than or equal to the equilibrium stress–strain modulus* $\tilde{\sigma}^*_\epsilon$, i.e.†

$$\tilde{E}(\epsilon^*, \eta^*, \tilde{\sigma}^*(\epsilon^*, \eta^*)) \geq \tilde{\sigma}^*_\epsilon(\epsilon^*, \eta^*) > 0. \tag{4.27}$$

Proof

Differentiating (3.18) with respect to stress and with respect to strain yield the relations

$$1 - \tilde{e}_{\sigma\epsilon} = \tilde{e}_{\sigma\sigma}\tilde{E} + \tilde{e}_\sigma \tilde{E}_\sigma, \tag{4.28}$$

$$-\tilde{e}_{\epsilon\epsilon} = \tilde{e}_{\sigma\epsilon}\tilde{E} + \tilde{e}_\sigma \tilde{E}_\epsilon. \tag{4.29}$$

Differentiating (4.21) and (4.24) with respect to strain implies

$$\tilde{\sigma}^*_\epsilon = \tilde{e}^*_{\epsilon\epsilon} = (\tilde{e}_{\epsilon\epsilon})^* + 2(\tilde{e}_{\sigma\epsilon})^*\tilde{\sigma}^*_\epsilon + (\tilde{e}_\sigma)^*\tilde{\sigma}^*_{\epsilon\epsilon} + (\tilde{e}_{\sigma\sigma})^*(\tilde{\sigma}^*_\epsilon)^2. \tag{4.30}$$

Evaluating (4.28) and (4.29) at equilibrium and using (4.30) and (4.15), we arrive at

$$\tilde{E}^* - \tilde{\sigma}^*_\epsilon = (\tilde{e}_{\sigma\sigma})^*(\tilde{E}^* - \tilde{\sigma}^*_\epsilon)^2$$

and the upper bound of (4.27) follows from (4.15). The lower bound is a consequence of (4.25), (4.26), and the relation

$$\tilde{\sigma}^*_\epsilon = -\frac{(\tilde{G}_\epsilon)^*}{(\tilde{G}_\sigma)^*},$$

which follows from (4.1) and (4.17).

Notice that (4.27), along with our assumption that $\tilde{E}$ never vanishes, is sufficient to guarantee that the instantaneous stress–strain modulus $\tilde{E}$ is strictly positive for all (ϵ, η, σ):

$$\tilde{E}(\epsilon, \eta, \sigma) > 0. \tag{4.31}$$

5. FURTHER RESTRICTIONS ON THE STRESS RELAXATION FUNCTION

Having introduced the concept of equilibrium, we can now examine further the consequences of the dissipation inequality (3.17). In particular, (3.17) imposes certain restrictions on the stress relaxation function $\tilde{G}$ and, following Malvern's suggestion[12] that $\tilde{G}$ be represented as a function of the overstress $\sigma - \tilde{\sigma}^*(\epsilon, \eta)$, we shall establish:‡

Theorem 6

Consider a generalized Maxwellian material whose instantaneous stress–strain modulus $\tilde{E}(\epsilon, \eta, \sigma)$ *is strictly positive. Further suppose that all equilibrium states of the material are asymptotically stable and that at these equilibrium states* $(\epsilon^*, \eta^*, \sigma^*)$

$$\begin{gathered} \tilde{e}_{\sigma\sigma}(\epsilon^*, \eta^*, \sigma^*) > 0, \\ \tilde{G}_\sigma(\epsilon^*, \eta^*, \sigma^*) < 0. \end{gathered} \tag{5.1}$$

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†We note that Greenberg[10] required the strict form of the inequality (4.27) to hold in order to prove the existence of steady waves in the context of the mechanical theory (1.1). Here we see that such an assumption has its foundatiions in thermodynamics. Coleman[11] has derived a similar result for materials with fading memory.

‡Since $\tilde{G}$ is a function of σ, ϵ, η, and the equilibrium function $\tilde{\sigma}^*$ is a function only of ϵ and η, we can always express the stress dependence of $\tilde{G}$ in terms of the overstress $\sigma - \tilde{\sigma}^*$. This representation is used quite frequently in applications.

Then, for every admissible strain-entropy pair (ϵ, η), *the stress relaxation function* $\tilde{G}(\epsilon, \eta, \sigma)$ *must be an odd function of the overstress* $\sigma - \tilde{\sigma}^*(\epsilon, \eta)$ *and satisfy*

$$\operatorname{sgn} \tilde{G}(\epsilon, \eta, \sigma) = -\operatorname{sgn}[\sigma - \tilde{\sigma}^*(\epsilon, \eta)]. \tag{5.2}$$

Proof

Using (3.18) and $\tilde{E} > 0$, the dissipation inequality (3.17) asserts that

$$[\sigma - \tilde{e}_\epsilon(\epsilon, \eta, \sigma)]\tilde{G}(\epsilon, \eta, \sigma) \leq 0 \tag{5.3}$$

Now let us construct the two C^1 functions

$$\tilde{\alpha}(\epsilon, \eta, \sigma) = \sigma - \tilde{e}_\epsilon(\epsilon, \eta, \sigma), \tag{5.4}$$

$$\tilde{\beta}(\epsilon, \eta, \sigma) = \sigma - \tilde{\sigma}^*(\epsilon, \eta). \tag{5.5}$$

It is clear from Theorem 4, that both α and β vanish at equilibrium and, in fact, as functions of σ, the functions $\tilde{\alpha}$ and $\tilde{\beta}$ must always vanish simultaneously. Furthermore, for the constant strain-entropy pair (ϵ^*, η^*), (5.4) and (5.5) imply that at equiilibrium

$$\tilde{\alpha}_\sigma(\epsilon^*, \eta^*, \sigma^*) = 1 - \tilde{e}_{\epsilon\sigma}(\epsilon^*, \eta^*, \sigma^*),$$
$$\tilde{\beta}_\sigma(\epsilon^*, \eta^*, \sigma^*) = 1.$$

But, by hypothesis, (4.28) and (5.1),

$$1 - (\tilde{e}_{\epsilon\sigma})^* = (\tilde{e}_{\sigma\sigma})^*(\tilde{E})^* > 0.$$

Thus, as functions of σ, both $\tilde{\alpha}$ and $\tilde{\beta}$ have slopes of the same sign at equilibrium. This fact, along with the observation that $\tilde{\alpha}$ and $\tilde{\beta}$ must always vanish simultaneously at equilibrium, implies that for every admissible strain-entropy pair (ϵ, η)

$$\operatorname{sgn}[\sigma - \tilde{e}_\epsilon(\epsilon, \eta, \sigma)] = \operatorname{sgn}[\sigma - \tilde{\sigma}^*(\epsilon, \eta)]$$

and hence, by (5.3),

$$[\sigma - \tilde{\sigma}^*(\epsilon, \eta)]\tilde{G}(\epsilon, \eta, \sigma) \leq 0. \tag{5.6}$$

This inequality yields the desired results.

The results in Theorem 6 suggest a possible definition for the *stress relaxation time* τ for a generalized Maxwellian material. In particular, we define this time by

$$\tau = \tilde{\tau}(\epsilon, \eta, \sigma) = -\frac{[\sigma - \tilde{\sigma}^*(\epsilon, \eta)]}{\tilde{G}(\epsilon, \eta, \sigma)}. \tag{5.7}$$

Then, the inequality (5.6) asserts that *the relaxation time* τ *is non-negative.* Notice that at equilibrium (5.7) implies

$$\tau^* = -\frac{1}{\tilde{G}_\sigma(\epsilon^*, \eta^*, \sigma^*)}$$

and thus, our assumption that the strict form of (4.16) holds (i.e. 5.1_2) is merely an assertion that the equilibrium value of the relaxation time τ^* is non-negative and finite.

6. CHARACTERIZATION OF INSTANTANEOUS RESPONSE

Another important aspect of the behavior of generalized Maxwellian materials concerns their response to small, but rapid, changes in strain and entropy. This response is elastic in nature and is termed the instantaneous response of the material. In particular, it is characterized by a function (see Greenberg[10])

$$\sigma = \tilde{\sigma}^+(\epsilon, \eta) \tag{6.1}$$

which has the properties

$$\tilde{\sigma}_\epsilon{}^+(\epsilon, \eta) = \tilde{E}(\epsilon, \eta, \tilde{\sigma}^+(\epsilon, \eta)), \tag{6.2}$$

$$\tilde{\sigma}_\eta{}^+(\epsilon, \eta) = \tilde{F}(\epsilon, \eta, \tilde{\sigma}^+(\epsilon, \eta)). \tag{6.3}$$

It is for this reason that we call $\tilde{E}$ and $\tilde{F}$ the instantaneous stress–strain modulus and the instantaneous stress-entropy modulus, respectively. In view of (6.1), (3.14) and (3.15) become

$$e = \tilde{e}(\epsilon, \eta, \tilde{\sigma}^+(\epsilon, \eta)) = \tilde{e}^+(\epsilon, \eta), \tag{6.4}$$

$$\theta = \tilde{\theta}(\epsilon, \eta, \tilde{\sigma}^+(\epsilon, \eta)) = \tilde{\theta}^+(\epsilon, \eta). \tag{6.5}$$

We call $\tilde{\sigma}^+$, $\tilde{e}^+$ and $\tilde{\theta}^+$ the *instantaneous response functions* for the material.

Using the chain-rule, (6.4) yields

$$\tilde{e}_\epsilon{}^+ = (\tilde{e}_\epsilon)^+ + (\tilde{e}_\sigma)^+ \tilde{\sigma}_\epsilon{}^+,$$
$$\tilde{e}_\eta{}^+ = (\tilde{e}_\eta)^+ + (\tilde{e}_\sigma)^+ \tilde{\sigma}_\eta{}^+,$$

and consequently, by (6.2), (6.3), (3.18), (3.19), (6.1) and (6.5), we have:

Theorem 7

The instantaneous response of a generalized Maxwellian material is described by the response function

$$e = \tilde{e}^+(\epsilon, \eta)$$

for all admissible pairs (ϵ, η), ***and the instantaneous stress*** σ ***and temperature*** θ ***are determined by the thermodynamic relations***

$$\tilde{\sigma}^+ = \tilde{e}_\epsilon{}^+, \quad \tilde{\theta}^+ = \tilde{e}_\eta{}^+. \tag{6.6}$$

7. A SPECIFIC CONSTITUTIVE MODEL

In the course of discussing the thermodynamics of generalized Maxwellian materials, we have established restrictions on the form of the constitutive equations. These restrictions are quite useful in formulating specific constitutive models. Here we wish to illustrate the formulation of such models by considering a simple example. To construct this example we must assume an explicit form for the dependence of the internal energy on the stress. We do this with the required form (3.14) and the minimal property (4.14) in mind. Consequently, we will assume that

$$e = \tilde{e}(\epsilon, \eta, \sigma) = \tilde{e}^*(\epsilon, \eta) + \frac{1}{2}\tilde{K}(\epsilon, \eta)[\sigma - \tilde{\sigma}^*(\epsilon, \eta)]^2 \tag{7.1}$$

where

$$\tilde{K}(\epsilon, \eta) > 0. \tag{7.2}$$

By noting (4.24) and Theorem 4, the derivatives of the internal energy function are computed to be

$$\tilde{e}_\epsilon = \tilde{\sigma}^* + \frac{1}{2}(\tilde{K}_\epsilon)(\sigma - \tilde{\sigma}^*)^2 - \tilde{K}(\sigma - \tilde{\sigma}^*)\tilde{\sigma}^*_\epsilon, \tag{7.3}$$

$$\tilde{e}_\eta = \tilde{\theta}^* + \frac{1}{2}(\tilde{K}_\eta)(\sigma - \tilde{\sigma}^*)^2 - \tilde{K}(\sigma - \tilde{\sigma}^*)\tilde{\sigma}^*_\eta, \tag{7.4}$$

$$\tilde{e}_\sigma = \tilde{K}(\sigma - \tilde{\sigma}^*). \tag{7.5}$$

We also require a definition of the explicit dependence of the temperature relation (3.15) on the stress and take

$$\theta = \tilde{\theta}(\epsilon, \eta, \sigma) = \tilde{\theta}^*(\epsilon, \eta) + \tilde{M}(\epsilon, \eta)(\sigma - \tilde{\sigma}^*(\epsilon, \eta)). \tag{7.6}$$

It is important to note that this relation must be the solution of (3.8) and, in view of (3.22) and (3.10),

$$\tilde{M}(\epsilon, \eta) \neq 0 \tag{7.7}$$

for all admissible pairs (ϵ, η). We now show that a relationship exists between $\tilde{M}$ and $\tilde{K}$. This relationship is derived by evaluating (7.1) at the instantaneous stress $\tilde{\sigma}^+(\epsilon, \eta)$ and invoking the relation $(6.6)_2$ to obtain

$$\tilde{\theta}^+ = \tilde{\theta}^* + \frac{1}{2}(\tilde{K}_\eta)(\tilde{\sigma}^+ - \tilde{\sigma}^*)^2 + \tilde{K}(\tilde{\sigma}^+ - \tilde{\sigma}^*)(\tilde{\sigma}_\eta{}^+ - \tilde{\sigma}^*_\eta). \tag{7.8}$$

If (7.6) is also evaluated for the instantaneous temperature, the result can be substituted into (7.8) to yield

$$\tilde{M} = \frac{1}{2}(\tilde{K}_\eta)(\tilde{\sigma}^+ - \tilde{\sigma}^*) + \tilde{K}(\tilde{\sigma}_\eta{}^+ - \tilde{\sigma}^*_\eta). \tag{7.9}$$

Notice that at a natural state; i.e. at a state where $\tilde{\sigma}^+ = \tilde{\sigma}^* = 0$, (7.9) and (7.2) assert that

$$\operatorname{sgn} \tilde{M} = \operatorname{sgn}(\tilde{\sigma}_\eta{}^+ - \tilde{\sigma}^*_\eta). \tag{7.10}$$

An additional constraint on K can be derived by evaluating (7.1) for the instantaneous stress and using $(6.6)_1$. This yields

$$\frac{1}{2}(\tilde{K}_\epsilon)(\tilde{\sigma}^+ - \tilde{\sigma}^*) + \tilde{K}(\tilde{\sigma}_\epsilon{}^+ - \tilde{\sigma}^*_\epsilon) - 1 = 0. \tag{7.11}$$

Then, using (7.3)–(7.6), (7.9) and (7.11), we can compute the moduli $\tilde{E}$ and $\tilde{F}$ from (3.18) and (3.19) as

$$\tilde{E} = \tilde{\sigma}_\epsilon{}^+ + \frac{1}{2}(\ln \tilde{K})_\epsilon(\tilde{\sigma}^+ - \sigma), \tag{7.12}$$

$$\tilde{F} = \tilde{\sigma}_\eta{}^+ + \frac{1}{2}(\ln \tilde{K})_\eta(\tilde{\sigma}^+ - \sigma). \tag{7.13}$$

Finally, we complete this example by specifying the stress relaxation function $\tilde{G}$ consistent with Theorem 6:

$$\tilde{G} = -\frac{1}{\tau}(\sigma - \tilde{\sigma}^*), \tag{7.14}$$

where $\tau > 0$ is a constant (see Malvern[12]).

It is of interest to note that, in connection with the mechanical theory (1.1), Schuler and Nunziato[13] have used the functional forms (7.12) and (7.14) for $\bar{E}$ and $\bar{G}$ to successfully describe the dynamic response of polymethyl methacrylate below 0.75 GPa.

8. AN ALTERNATIVE FORMULATION FOR GENERALIZED MAXWELLIAN MATERIALS

In the theory of generalized Maxwellian materials we have shown that the stress acts very much like an internal state variable. Here we wish to point out that the theory could have been alternatively formulated so that the temperature acts as the internal state variable.

Returning to Section 3, we note that by (3.10) and (3.22) the function $\tilde{\theta}$ is invertible in σ for every admissible pair (ϵ, η). Consequently, there exists a function $\bar{\sigma}$;

$$\sigma = \bar{\sigma}(\epsilon, \eta, \theta), \tag{8.1}$$

which is equivalent to (2.5). This equivalence leads directly to the following analog of Theorem 2.

Theorem 8

A generalized Maxwellian material can be alternatively characterized by the constitutive equations

$$e = \bar{e}(\epsilon, \eta, \theta) \tag{8.2}$$

$$\sigma = \bar{\sigma}(\epsilon, \eta, \theta) \tag{8.3}$$

$$\dot{\theta} = \bar{A}(\epsilon, \eta, \theta)\dot{\epsilon} + \bar{B}(\epsilon, \eta, \theta)\dot{\eta} + \bar{C}(\epsilon, \eta, \theta) \tag{8.4}$$

subject to the dissipation inequality

$$\bar{e}_\theta(\epsilon, \eta, \theta)\bar{C}(\epsilon, \eta, \theta) \leq 0. \tag{8.5}$$

Furthermore, the response functions $\bar{A}$ and $\bar{B}$ are not arbitrary but are determined by $\bar{e}$ and $\bar{\sigma}$ through the relations

$$\bar{A}(\epsilon, \eta, \theta)\bar{e}_\theta(\epsilon, \eta, \theta) = \bar{\sigma}(\epsilon, \eta, \theta) - \bar{e}_\epsilon(\epsilon, \eta, \theta), \tag{8.6}$$

$$\bar{B}(\epsilon, \eta, \theta)\bar{e}_\theta(\epsilon, \eta, \theta) = \theta - \bar{e}_\eta(\epsilon, \eta, \theta). \tag{8.7}$$

It should be apparent that this formulation of the theory can now be developed along the same lines as those followed in Sections 4–6. Of course, the instantaneous and equilibrium response functions in the present context will be the same as those already defined. However, we shall not take up the details of this analysis here.

In closing, we wish to merely emphasize the ease by which one can go from the formulation in terms of stress in Theorem 2 to the formulation in terms of temperature in Theorem 8. In particular, if the functions

$$e = \tilde{e}(\epsilon, \eta, \sigma), \tag{8.8}$$

$$\theta = \tilde{\theta}(\epsilon, \eta, \sigma), \tag{8.9}$$

$$G = \tilde{G}(\epsilon, \eta, \sigma) \tag{8.10}$$

are given, we can then invert (8.9) to get (8.3). Hence, it follows from (8.8) and (8.10) that

$$e = \tilde{e}(\epsilon, \eta, \bar{\sigma}(\epsilon, \eta, \theta)) = \bar{e}(\epsilon, \eta, \theta),$$
$$G = \tilde{G}(\epsilon, \eta, \bar{\sigma}(\epsilon, \eta, \theta)) = \bar{G}(\epsilon, \eta, \theta).$$

Furthermore, the moduli $\bar{A}$ and $\bar{B}$ can be computed according to (8.6) and (8.7), and $\bar{C}$ can be deduced from (3.22):

$$\bar{\bar{C}}(\epsilon, \eta, \theta) = \frac{\bar{\bar{G}}(\epsilon, \eta, \theta)}{\bar{\bar{\sigma}}_\theta(\epsilon, \eta, \theta)}. \tag{8.11}$$

We note that in this formulation the energy equation has the form

$$\theta\dot{\eta} + \bar{\bar{e}}_\theta(\epsilon, \eta, \theta)\bar{\bar{C}}(\epsilon, \eta, \theta) = r. \tag{8.12}$$

9. CLOSING REMARKS

In this paper we have presented a complete thermomechanical formulation for Maxwellian materials which do not conduct heat. As one might expect we have shown that the internal energy function provides a complete thermodymamical description of the material only when the response is elastic, i.e. at the equilibrium and instantaneous states. For other states additional response functions must be specified. In the context of the stress-rate formulation, a temperature response function and a stress-relaxation function must be specified; and, in particular, the stress-relaxation function must be specified as an odd function of the overstress.

We have also shown that, in one dimension, two formulations of the response functions are possible, and that these two choices are entirely equivalent formulations. The stress-rate formulation is commonly used in practical problems; however, the temperature-rate formulation is possibly more useful since it involves a scalar rate-relation for modeling bulk relaxation phenomena.

Acknowledgements—We would like to express our gratitude to R. M. Bowen and C. Truesdell for their helpful comments on a previous draft of this manuscript. This work was supported by the U.S. Department of Energy under contract AT(29–1)789.

Note added in proof. Following the completion of this work, we had the opportunity to share our results with M. E. Gurtin of Carnegie-Mellon University. At that time, he indicated that he had discussed similar ideas with Ion Suliciu of the Institute of Mathematics, Bucharest, Romania in 1972 and the result was an unpublished manuscript containing results similar to our Theorems 1 and 4. Suliciu subsequently included their results in a book[14] co-authored with N. Cristescu.

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